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RELAXATION FROM STEADY STATES FAR FROM EQUILIBRIUM AND THE PERSISTENCE OF ANOMALOUS SHOCK BEHAVIOR IN WEAKLY IONIZED GASES

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Abstract. The decay of anomalous effects on shock waves in weakly ionized gases following plasma generator extinction has been measured in the anticipation that the decay time must correlate well with the relaxation time of the mechanism responsible for the anomalous effects. When the relaxation times cannot be measured directly, they are inferred theoretically, usually assuming that the initial state is nearly in thermal equilibrium. In this paper, it is demonstrated that relaxation from any steady state far from equilibrium, including the state of a weakly ionized gas, can proceed much more slowly than arguments based on relaxation from near equilibrium states might suggest. This result justifies a more careful analysis of the relaxation times in weakly ionized gases and suggests that although the experimental measurements of relaxation times did not lead to an unambiguous conclusion, this approach to understanding the anomalous effects may warrant further investigation.

Key words. weakly ionized gases, drag reduction

Subject classification. Physical Sciences

1. Introduction. Renewed attention has recently been given to experimental reports of anomalous behavior of shock waves in weakly ionized gases [1]. In an attempt [2] to isolate the mechanism responsible for these effects, shocks were measured in a decaying plasma following extinction of the plasma generator. It was argued that the measured relaxation time of the anomalous effects must correlate well with the the relaxation time of their unknown cause; this measurement might therefore suggest the mechanism responsible for the anomalous effects, or perhaps even identify it unambiguously.

Relaxation times cannot always be measured directly; in such cases, they are inferred theoretically, usually by assuming relaxation from a near-equilibrium state. But since a two-temperature weakly ionized gas in maintained in a steady state by energizing the electrons against the energy sink provided by heat conduction [3], such a weakly ionized gas is really in a steady state far from equilibrium. By analyzing representative examples, the present work shows that systems which are initially far from equilibrium relax to equilibrium much more slowly than systems which are initially near equilibrium.

Retarded relaxation is demonstrated analytically for two systems: the Kats-Kontorovich steady state [4], a steady state maintained by sources and sinks of particles and energy in a gas of particles interacting through a power-law potential, and a nonequilibrium state of a gas of light particles diffusing in a gas of heavy particles [5]. In particular, relaxation of the Kats-Kontorovich steady state following extinction of the sources and sinks is shown to be algebraic in time instead of exponential: the relaxation rate is computed theoretically to be $1/t$ where t is time.

The existence of non-exponential relaxation in systems far from equilibrium suggests that relaxation times in weakly ionized gases should not be computed based on relaxation from a near-equilibrium state.

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This result justifies a more careful analysis of the relaxation times in weakly ionized gases and suggests that although the experimental measurements of relaxation times did not lead to an unambiguous conclusion, this approach to understanding the anomalous effects may warrant further investigation.

2. Steady solutions of the kinetic equations. The H-theorem of statistical mechanics [5] demonstrates that the Maxwell-Boltzmann distribution is the unique long-time limit of the distribution function of any isolated many particle system. If the system is near but not in thermal equilibrium, then the Chapman-Enskog expansion shows [6] that the Boltzmann equation

$$\frac{\partial f}{\partial t} = \Omega(f, f) \tag{2.1}$$

can be approximated by the BGK equation

$$\frac{\partial f}{\partial t} = -\omega \{f - f_{eq}\}. \tag{2.2}$$

In Eq. (2.1), $f = f(\mathbf{r}, \mathbf{p}, t)$ is the single-particle distribution function and $\Omega(f, f)$ is the Boltzmann collision operator [5]. In Eq. (2.2), f_{eq} denotes the Maxwell-Boltzmann distribution function; this equation states that a near-equilibrium distribution function relaxes exponentially to a Maxwellian, where the relaxation time ω^{-1} is given[6] for a hard-sphere gas by

$$\omega \sim \left.\frac{\delta\Omega}{\delta f}\right|_{f=f_{eq}} = \frac{\rho d^2}{4m}\sqrt{\frac{T}{m}}. \tag{2.3}$$

In Eq. (2.3), ρ is the mass density of the gas, m is the mass of a gas particle, d is the particle diameter, and T is the temperature; units in which the Boltzmann constant $k = 1$ are assumed.

Analogous exponential relaxation is directly evident in the Fokker-Planck equation[5] for a gas of light particles diffusing in a gas of heavy particles,

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{1}{p^2}\frac{\partial(p^2 s)}{\partial p} \\ s &= -B\left\{\frac{p}{mT}f + \frac{\partial f}{\partial p}\right\}. \end{aligned} \tag{2.4}$$

In Eq. (2.4), m and p are the mass and momentum of a light particle and T is the temperature of the gas of heavy particles. In this problem, the Maxwellian distribution is recovered by setting the flux $s = 0$; relaxation to this state from a nearby state occurs over a time of order ω^{-1} with

$$\omega \sim \frac{B}{mT}. \tag{2.5}$$

Although the H-theorem determines the long-time limit of the Boltzmann equation, it does not dictate the time required to reach this limit. We will consider initial states from which relaxation to equilibrium is particularly slow. Both the Boltzmann equation [4] and Fokker-Planck equation admit solutions representing non-Maxwellian steady states far from equilibrium: these are solutions of the steady forms of Eqs. (2.1) and (2.4) which prove to be singular at zero or infinite momentum. Thus, the non-Maxwellian solutions of Eq. (2.1) are not integrable over all momenta: it will be shown that the integral representing the total number of particles diverges at infinite momentum. Although this divergence might seem to be of merely formal interest, it is actually another significant consequence of the H-theorem, which rules out any steady solution of Eq. (2.1) other than the Maxwellian. Similar singularities will be shown to exist for the non-Maxwellian solution of the Fokker-Planck equation Eq. (2.4).

It follows that these distributions only exist in a finite region of momentum space. If they are extended to be zero outside this region, the result is a solution of the *inhomogeneous* steady Boltzmann or Fokker-Planck equation

$$\Omega(f,f) = -\dot{S} \tag{2.6}$$

$$-\frac{1}{p^2}\frac{\partial(p^2 s)}{\partial p} = -\dot{S} \tag{2.7}$$

where the source term $\dot{S}$ is a sum of delta functions.

Since the source $\dot{S}$ in Eqs. (2.6) and (2.7) vanishes except over a finite number of spheres in momentum space, the far from equilibrium distributions satisfy homogeneous equations everywhere else. From this viewpoint, the slow relaxation of these states is immediately suggested by the observation that initially, $\partial f/\partial t = 0$ except where the source is non-zero. Relaxation of the non-equilibrium distribution requires that the effects of removing the source diffuse over all of momentum space; this process can be very slow.

3. The Kats-Kontorovich steady state. The collision integral is given in terms of the transition probability $U(\mathbf{p}_1', \mathbf{p}', \mathbf{p}_1, \mathbf{p})$, where $\mathbf{p}_1', \mathbf{p}'$ are post-collision and $\mathbf{p}_1, \mathbf{p}$ are pre-collision momenta, by

$$\begin{aligned}\Omega(f,f) = \frac{d^2}{m^2}\int d\mathbf{p}_1' d\mathbf{p}' d\mathbf{p}_1 \delta(E_1' + E' - E_1 - E) \times \\ \delta(\mathbf{p}_1' + \mathbf{p}' - \mathbf{p}_1 - \mathbf{p}) U(\mathbf{p}_1', \mathbf{p}', \mathbf{p}_1, \mathbf{p})\{f'f_1' - ff_1\}\end{aligned} \tag{3.1}$$

where f, f_1 are pre-collision and f', f_1' are post-collision distribution functions,

$$\begin{aligned} f &= f(\mathbf{r}, \mathbf{p}, t) \\ f_1 &= f(\mathbf{r}, \mathbf{p}_1, t) \\ f' &= f(\mathbf{r}, \mathbf{p}', t) \\ f_1' &= f(\mathbf{r}, \mathbf{p}_1', t)\end{aligned} \tag{3.2}$$

and E is the kinetic energy

$$E(p) = \frac{p^2}{2m}. \tag{3.3}$$

The conservation laws of particle mechanics imply that for any distribution function f,

$$\begin{aligned} \int d\mathbf{p}\ \Omega(f,f) &= 0 \\ \int \mathbf{p} d\mathbf{p}\ \Omega(f,f) &= 0 \\ \int E(p) d\mathbf{p}\ \Omega(f,f) &= 0.\end{aligned} \tag{3.4}$$

For *non-equilibrium* f, non-zero fluxes in momentum space

$$\begin{aligned} J_m(p') &= -\int_{p \geq p'} d\mathbf{p}\ \Omega(f,f) \\ J_e(p') &= -\int_{p \geq p'} d\mathbf{p}\ E(p)\Omega(f,f)\end{aligned} \tag{3.5}$$

are possible, where J_m is a particle flux, and J_e is an energy flux. Thus, if

$$\dot{S}(p_0) = A_0\delta(p-p_0) + B_0\delta'(p-p_0), \tag{3.6}$$

then the corresponding fluxes are

$$J_m = 4\pi \begin{cases} 0 & \text{if } p \le p_0 \\ A_0p_0^2 - 2B_0p_0^3 & \text{if } p \ge p_0. \end{cases} \tag{3.7}$$

and

$$J_e = \frac{4\pi}{2m} \begin{cases} 0 & \text{if } p \le p_0 \\ A_0p_0^4 - 4B_0p_0^5 & \text{if } p \ge p_0. \end{cases} \tag{3.8}$$

It follows that the source distribution

$$\dot{S} = \frac{2J}{4\pi}\{\frac{1}{p_0^2}\delta(p-p_0) + \frac{1}{4p_0}\delta'(p-p_0) - \frac{1}{p_1^2}\delta(p-p_1) - \frac{1}{4p_1}\delta'(p-p_1)\} \tag{3.9}$$

satisfies the equations

$$\begin{aligned} \int d\mathbf{p}\ \dot{S} &= 0 \\ \int \mathbf{p}d\mathbf{p}\ \dot{S} &= 0 \\ \int E(p)d\mathbf{p}\ \dot{S} &= 0. \end{aligned} \tag{3.10}$$

In view of Eqs. (2.6) and (3.4), Eq. (3.10) implies that the source term $\dot{S}$ does not alter the total mass, momentum, or energy of the system; a steady state with $\dot{S}$ as a source term is therefore possible. The fluxes are

$$\begin{aligned} J_m &= \begin{cases} 0 & \text{if } p \le p_0 \\ J & \text{if } p_0 \le p \le p_1 \\ 0 & \text{if } p \ge p_1 \end{cases} \\ J_e &\equiv 0. \end{aligned} \tag{3.11}$$

Similarly,

$$\dot{S} = \frac{-J'}{8m\pi}\{\frac{1}{p_0^4}\delta(p-p_0) + \frac{1}{2p_0^3}\delta'(p-p_0) - \frac{1}{p_1^4}\delta(p-p_1) - \frac{1}{2p_1^3}\delta'(p-p_1)\} \tag{3.12}$$

satisfies Eq. (3.10) and yields fluxes

$$\begin{aligned} J_e &= \begin{cases} 0 & \text{if } p \le p_0 \\ J' & \text{if } p_0 \le p \le p_1 \\ 0 & \text{if } p \ge p_1 \end{cases} \\ J_m &\equiv 0. \end{aligned} \tag{3.13}$$

An alternative formulation [7] is to consider the Boltzmann equation with external forcing

$$\mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \Omega(f, f). \tag{3.14}$$

At any discontinuity of the distribution function f, a delta function singularity occurs. The singular source distribution $\dot{S}$ could therefore be replaced by an appropriately chosen external force; however, we will not pursue this formulation further.

Kats *et al.* [4] show that if U is homogeneous of degree μ,

$$U(\lambda\mathbf{p}_1', \lambda\mathbf{p}', \lambda\mathbf{p}_1, \lambda\mathbf{p}) = \lambda^{\mu} U(\mathbf{p}_1', \mathbf{p}', \mathbf{p}_1, \mathbf{p}) \tag{3.15}$$

then the collision integral Eq. (3.1) vanishes[4] for the power-law distribution

$$f(\mathbf{p}) = Cm^{1/2} d^{-1} p_*^{\mu/2} \mid J \mid^{1/2} p^{-7/2-\mu/2} \tag{3.16}$$

corresponding to a steady state with constant mass flux $J = J_m$. In Eq. (3.16), p_* is defined by the requirement that

$$U(\mathbf{p}_1', \mathbf{p}', \mathbf{p}_1, \mathbf{p}) = (\frac{p}{p_*})^{\mu} U_0(\mathbf{p}_1', \mathbf{p}', \mathbf{p}_1, \mathbf{p}) \tag{3.17}$$

with U_0 homogeneous of degree zero. They also demonstrate that the distribution Eq. (3.16) is *local*, in the sense that the collision integral converges when evaluated on this distribution taken over all momenta, provided

$$-3 < \mu < -1 \tag{3.18}$$

and the power law exponent is in the range

$$-3 < -\frac{7}{2} - \frac{\mu}{2} < -2. \tag{3.19}$$

It follows from Eq. (3.19) that the integrals representing the total number of particles and the total energy

$$\begin{aligned} N &= \int d\mathbf{p} f(\mathbf{p}) \\ E &= \int E(p) d\mathbf{p} f(\mathbf{p}) \end{aligned} \tag{3.20}$$

both diverge at large momentum; therefore, the steady state distribution function Eq. (3.16) cannot exist for all momenta. The distribution function must be cut off at some finite value $p = p_1$. If a low momentum cutoff $p = p_0$ is assumed as well, so that Eq. (3.16) applies only when $p_0 \leq p \leq p_1$, then this trucated distribution with fluxes of the form Eq. (3.11) is the solution of the inhomogeneous Boltzmann equation with a source of the form Eq. (3.9).

4. Relaxation of the Kats-Kontorovich steady state. We will investigate the relaxation of the steady state Eq. (3.16) to thermal equilibrium following removal of the sources and sinks. It will be convenient to work with an infinite power-law distribution in which the number of particles is infinite. For the Kats-Kontorovich distribution cut off at momentum p_1, the limiting Maxwellian distribution satisfies the inequality

$$f_0 \leq C \frac{\rho}{T^{3/2}} \tag{4.1}$$

where C is a constant. The mass density and temperature scale in terms of the limiting momentum p_1 as

$$\begin{aligned} \rho &\sim N \sim p_1^{-(\mu+1)/2} \\ T &\sim E \sim p_1^{-(\mu-3)/2} \end{aligned} \tag{4.2}$$

therefore

$$f_0 \sim p_1^{(\mu-11)/4}. \tag{4.3}$$

It follows that in the limit $p_1 \to \infty$, the limiting Maxwellian distribution vanishes pointwise, although its integral over all momenta becomes infinite. The infinite region of integration makes this combination of limits possible.

The initial relaxation rate of the Kats-Kontorovich distribution is

$$\omega = \frac{\delta\Omega}{\delta f} = \frac{d}{m^{1/2}} J_0^{1/2} p^{(\mu+1)/2} p_*^{-\mu/2} \tag{4.4}$$

where the functional derivative is evaluated at the Kats-Kontorovich distribution. The absolute value of the flux is written as J_0 to indicate that this is its initial value; the particle flux will become momentum dependent and will decay with time as the distribution relaxes to equilibrium.

Eq. (4.4) suggests that we seek a similarity solution of the time-dependent Boltzmann equation Eq. (2.1) of the form

$$f(\mathbf{p},t) = Cm^{1/2} d^{-1} J_0^{1/2} p^{-7/2-\mu/2} \phi\left(\frac{d}{m^{1/2}} p^{(\mu+1)/2} p_*^{-\mu/2} t J^{1/2}\right). \tag{4.5}$$

Introduce the substitution

$$q = \frac{d}{m^{1/2}} p^{(\mu+1)/2} p_*^{-\mu/2} t J^{1/2} \tag{4.6}$$

in the collision integral. The result is the equation

$$\begin{aligned}
&\dot{\phi} = q^{6/(\mu+1)} \int (q_1' q' q_1)^{6/(\mu+1)-1} d\mathbf{o}_1 d\mathbf{o}' d\mathbf{o}_1' dq_1' dq' dq_1 \times \\
&\delta[(q_1')^{4/(\mu+1)} e_1' + (q')^{4/(\mu+1)} e' - (q_1)^{4/(\mu+1)} e_1 - (q)^{4/(\mu+1)} e] \times \\
&\delta[(q_1')^{2/(\mu+1)} \mathbf{o}_1' + (q')^{2/(\mu+1)} \mathbf{o}' - (q_1)^{2/(\mu+1)} \mathbf{o}_1 - (q)^{2/(\mu+1)} \mathbf{o} \times \\
&U[(q_1')^{2/(\mu+1)} \mathbf{o}_1', (q')^{2/(\mu+1)} \mathbf{o}', (q_1)^{2/(\mu+1)} \mathbf{o}_1, (q)^{2/(\mu+1)} \mathbf{o}] \times \\
&[(q_1')^{-(7+\mu)/(1+\mu)} \phi(q_1') (q')^{-(7+\mu)/(1+\mu)} \phi(q') \\
&-(q_1)^{-(7+\mu)/(1+m)} \phi(q_1) (q)^{-(7+\mu)/(1+m)} \phi(q)]
\end{aligned} \tag{4.7}$$

where locality property has been assumed in order to integrate over all momenta. Locality of the relaxating distribution will be demonstrated later for a restricted set of exponents μ.

If we write Eq. (4.7) simply as

$$\dot{\phi}(q) = \int \mathsf{K}(q_1', q', q_1, q)[\phi(q_1')\phi(q') - \phi(q_1)\phi(q)] dq_1' dq' dq_1 \tag{4.8}$$

then power counting shows that

$$\mathsf{K}(\lambda q_1', \lambda q', \lambda q_1, \lambda q) = \lambda^\alpha \mathsf{K}(q_1', q', q_1, q) \tag{4.9}$$

where

$$\alpha = -3. \tag{4.10}$$

This homogeneity immediately implies that the integral equation Eq. (4.8) admits a solution

$$\phi(q) = Cq^{-1}. \tag{4.11}$$

The relaxing distribution function has therefore the asymptotic form

$$f = \frac{C}{t} p^{-4-\mu} p_*^{\mu} \tag{4.12}$$

and the flux decays according to the power law

$$J \sim t^{-2}. \tag{4.13}$$

The conclusion that the distribution function relaxes algebraically with time is in sharp contrast to the exponential relaxation of a near-equlibrium distribution.

These calculations all require the convergence of the relevant collision integrals. Therefore, both the original Kats-Kontorovich distribution and the relaxing distribution Eq. (4.12) must be local. The results of Ref. 4 show that both distributions are local provided $-3 < \mu < -2$; for the distributions with $-2 < \mu < -1$, the relaxing distribution is nonlocal, and the calculation leading to Eq. (4.12) is incomplete.

5. Constant flux solution of the Fokker-Planck equation. The Fokker-Planck equation for a gas of light particles diffusing in a gas of heavy particles is given by Eq. (2.4). The thermal equilibrium solution is found by setting the flux $s = 0$, leading to the Maxwell-Boltzmann distribution

$$f_{eq} = a \exp(-\frac{p^2}{2mT}) \tag{5.1}$$

A steady state far from equilibrium is defined by Eq. (2.4) by making the flux s non-zero; the corresponding distribution function is

$$f(p) = \frac{\dot{\rho}_0}{m} \exp(-\frac{p^2}{2mT}) \int_0^p dp \ \exp(\frac{p^2}{2mT}) + A f_{eq} \tag{5.2}$$

where A is any constant.

For large momentum, this distribution function has the asymptotic expansion

$$f(p) \sim \frac{(mT)}{p^3} + \frac{3(mT)^2}{p^5} + \cdots \tag{5.3}$$

Evaluating Eqs. (3.20) at large momentum,

$$\begin{aligned} &N \sim \log p \quad p \to \infty \\ &E \sim p^2 \quad p \to \infty. \end{aligned} \tag{5.4}$$

Also, the second part of Eq. (2.4) implies that the flux s is singular when $p = 0$. Therefore, like the Kats-Kontorovich steady state, the far from equilibrium distribution described by Eq. (5.2) only exists in a finite region $p_0 \leq p \leq p_1$ and must be maintained by suitable sources and sinks.

If the sources and sinks are removed, the distribution relaxes to Maxwellian. The relaxation time is simply the time required for the effect of shutting down the sources at p_0, p_1 to diffuse over all of momentum space. Since the equation is linear, this time can be estimated immediately as $(p_1 - p_0)^2/B$ instead of T/B as predicted by Eq. (2.5) for relaxation of a near-equilibrium distribution. Thus, the time required for the nonequilibrium distribution to relax to equilibrium depends on the momentum range of the distribution and is generally much larger than the time to relax a near-equilibrium distribution. Note also that, as in the Kats-Kontorovich distribution, the large momentum tail contains considerable energy in view of the divergence in Eq. (5.4).

In comparing the results of Sects. 4 and 5, it is apparent that the requirement of relaxation of the effects of source extinction is common to both problems, but relaxation from the Kats-Kontorovich steady state is further impeded by the slow decay of the flux J. In the Fokker-Planck equation, the relaxation rate is always proportional to B/T. It is independent of the distribution function of the light particles because the derivation of the equation assumes that the distribution function of the heavy particles is not altered by the light particles.

6. Application to shocks in weakly ionized gases. Any explanation of the reported anomalous properties of shock waves in weakly ionized gases must be advanced with caution in view of the current unsettled status of the supporting experimental observations. At least some of these observations can be explained as consequences of temperature inhomogeneities in the experiments [8], [9]; from this viewpoint, the reported anomalies do not reveal any unexpected or remarkable consequences of the statistical mechanics of the weakly ionized gases. Further experimental progress [9] will be required to resolve these questions.

If the reports of anomalous behavior are accepted tentatively, then the present results suggest that although the measurements reported by Klimov *et al.* [2] did not lead as anticipated to the cause of the anomalous effects, the approach based on relaxation times might have failed because some of the relaxation times were computed assuming relaxation from a near-equilibrium state. Further investigation of the relaxation mechanisms in a weakly ionized gas should account for the fact that the initial state is far from thermal equilibrium.

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